Z-SCORES

Objectives

1. You will learn how z-scores can be used to evaluate how extreme a given value is in a particular set or population.
2. You will be able to evaluate z-scores as they relate to probability.
3. You will be able to calculate the probability of a z-score between two others.
4. You will be able to find the z-score, given the probability.

Say I had a distribution with a mean $\mu = 55$ and a standard deviation $\sigma = 12$ ...

How many standard deviations from the mean is the value $x = 67$? 1

This number is called the z-score

How many standard deviations from the mean is the value $x = 79$? 2

How many standard deviations from the mean is the value $x = 73$? 1.5

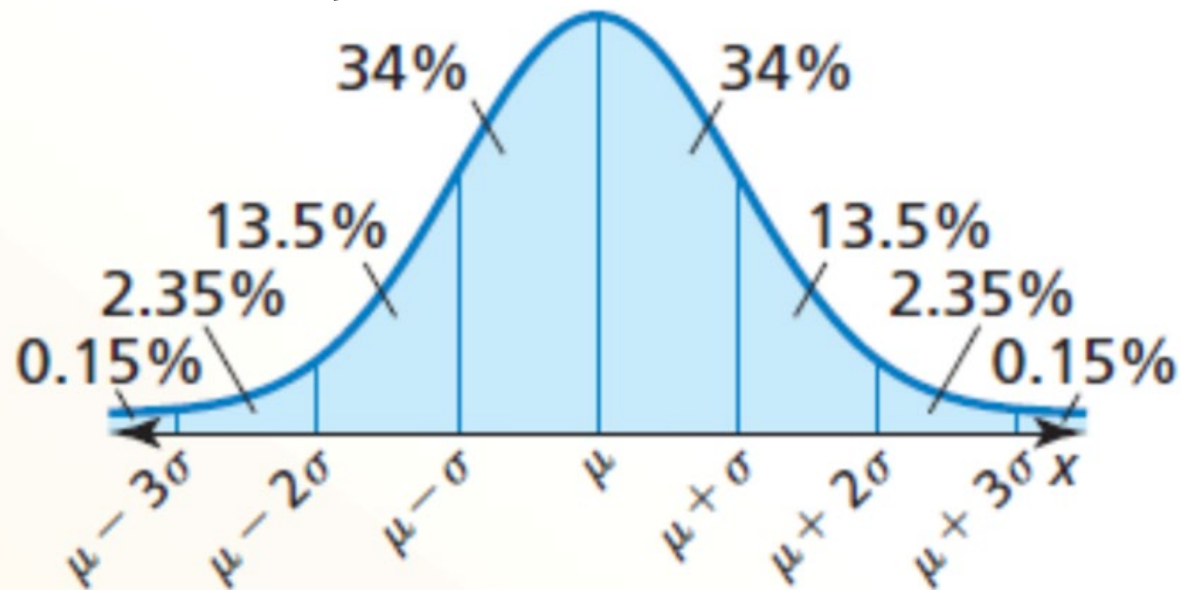How many standard deviations from the mean is the value $x = 70$? 1.25

Using the Empirical Rule can give you a good idea of the probability of occurrence of a value that happens to be exactly one, two or three standard deviations to either side of the mean, but how do you compare the probabilities of values that are in between those standard deviations?

The z-score of a value is the number of standard deviations between the value and the mean of the set.

Conceptually, the z-score calculation is just what you might expect, given that you are calculating the number of SDs between a value and the mean. You calculate the z-score by first calculating the difference between your value and the mean, and then dividing that amount by the standard deviation of the set. The formula looks like this:

$$\text{z-score} = \frac{(\text{value} - \text{mean})}{\text{standard deviation}} = \frac{(x - \mu)}{\sigma}$$

μ (lower case mu) = mean

σ (lower case sigma) = standard deviation
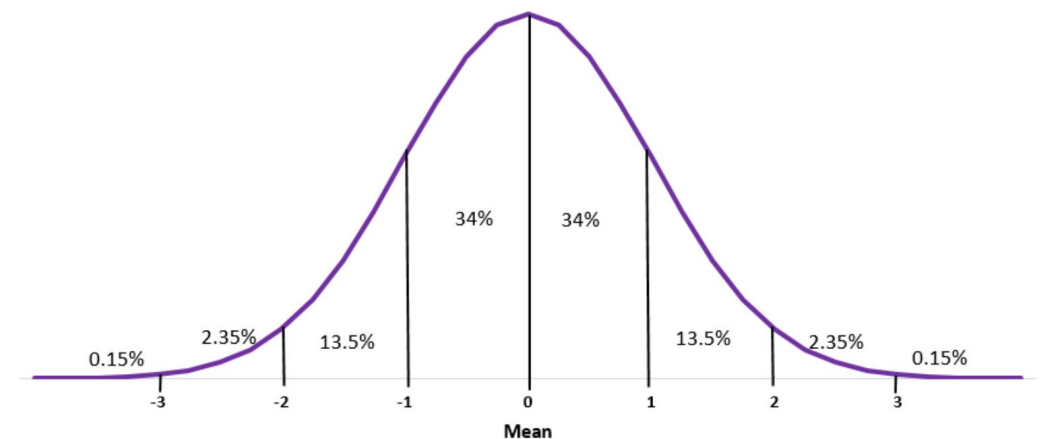
What is the *z*-score of a value of 27, given a sample mean of 24, and a standard deviation of 2?

SOLUTION

To find the *z*-score we need to divide the difference between the sample mean value, and the mean, 24, by the standard deviation, 2

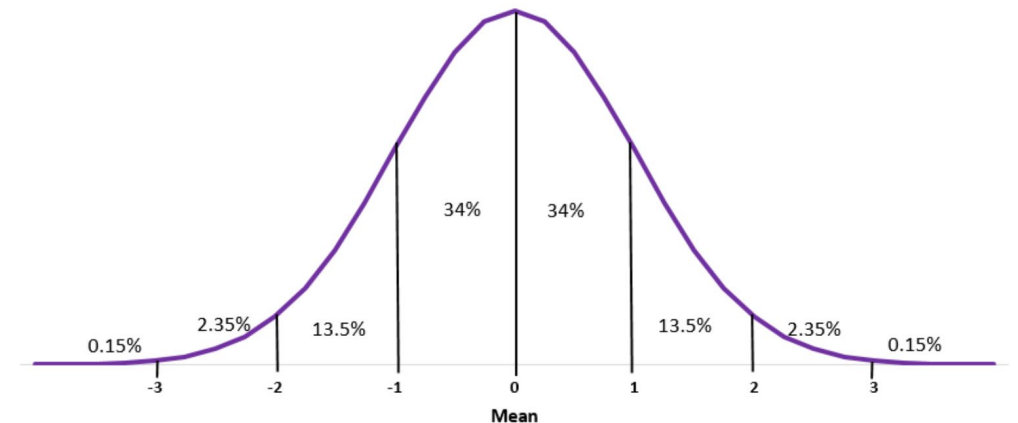$$z = \frac{27-24}{2} = \frac{3}{2}$$

This indicates that 27 is 1.5 standard deviations above the mean

On your own: What is the z-score of a value, x, of 104.5, in a population with a mean $\mu = 125$ and a standard deviation $\sigma = 6.2$?

$$z = \frac{\bar{x}-\mu}{\sigma} = \frac{104.5-125}{6.2} = -3.306$$

Note that the z-score is <u>negative</u> because $x = 104.5$ is <u>less</u> than the mean $\mu = 125$.

Find the value x represented by a z-score of 2.403, given $\mu = 63$ and $\sigma = 4.25$

This one requires that we solve for a missing value rather than for a missing z-score, so we just need to fill in our formula with what we know and solve for the missing value.

$$z = \frac{x - \mu}{\sigma}$$

$$2.403 = \frac{x - 63}{4.25}$$

$$10.213 = x - 63$$

$$73.213 = x$$

73.213 has a z-score of 2.403

What is the z-score of the price of a pair of skis that cost $247, if the mean ski price is $279 with a standard deviation of $16?

$$z = \frac{247 - 279}{16} = \frac{-32}{16} = -2$$

Since z-scores are a measure of the number of standard deviations (SDs) between a value and the mean, they can be used to calculate probability by comparing the location of the z-score to the area under a normal curve either to the left or right. The area can be calculated using calculus, but we will just use a table to look up the area.
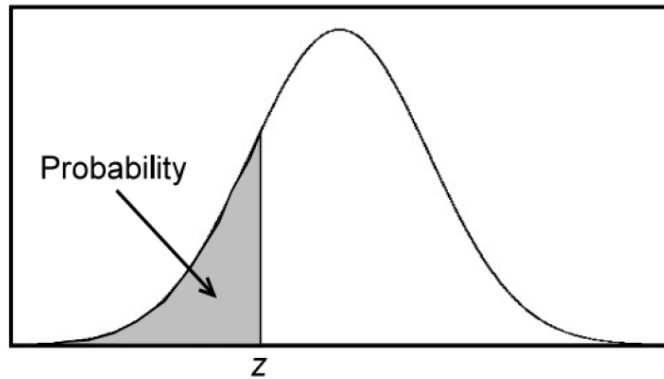

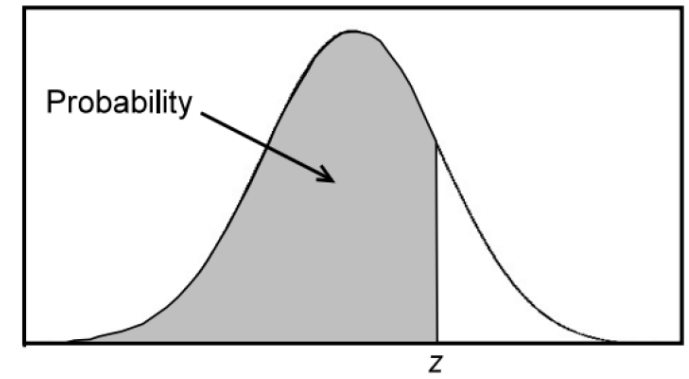Table entry for z is the probability lying below z.

Table entry for z is the probability lying below z.

Table A Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084

Table A (Continued)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5399
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830

z-score = # of σ from the μ

The Empirical Rule only provides probability percentages for **INTEGER** values of z-scores (standard deviations). To find the values for z-scores that aren't integers we use a z-score table.

To find the value associated with a given z-score, find the first decimal (tenths place) for your z-score in the left column then the second decimal (hundredths place) across the top of the table. Where they intersect is the decimal expression of the % of values that are **LESS THAN** your sample.
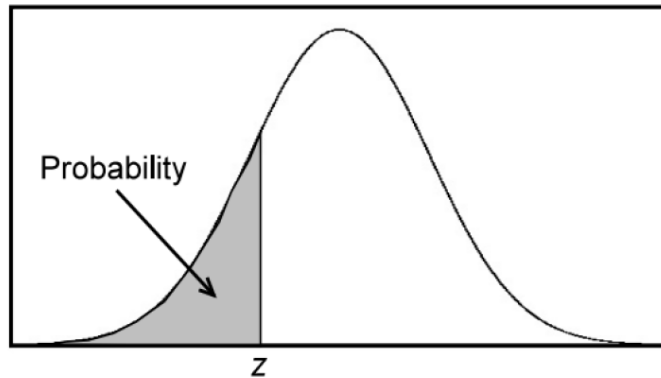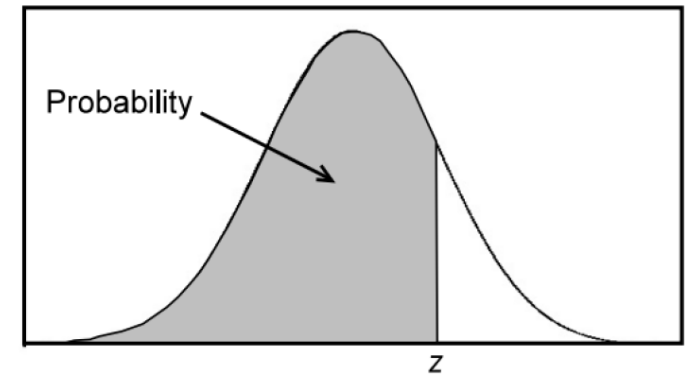
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−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019

Table A   (Continued)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5399
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549

1. What is the probability of occurrence of a value with a z-score less than 1.24?

We use the value for z = 1.24 directly from the table, and just express it as a percentage: 0.8925 or 89.25%

Table A   (Continued)

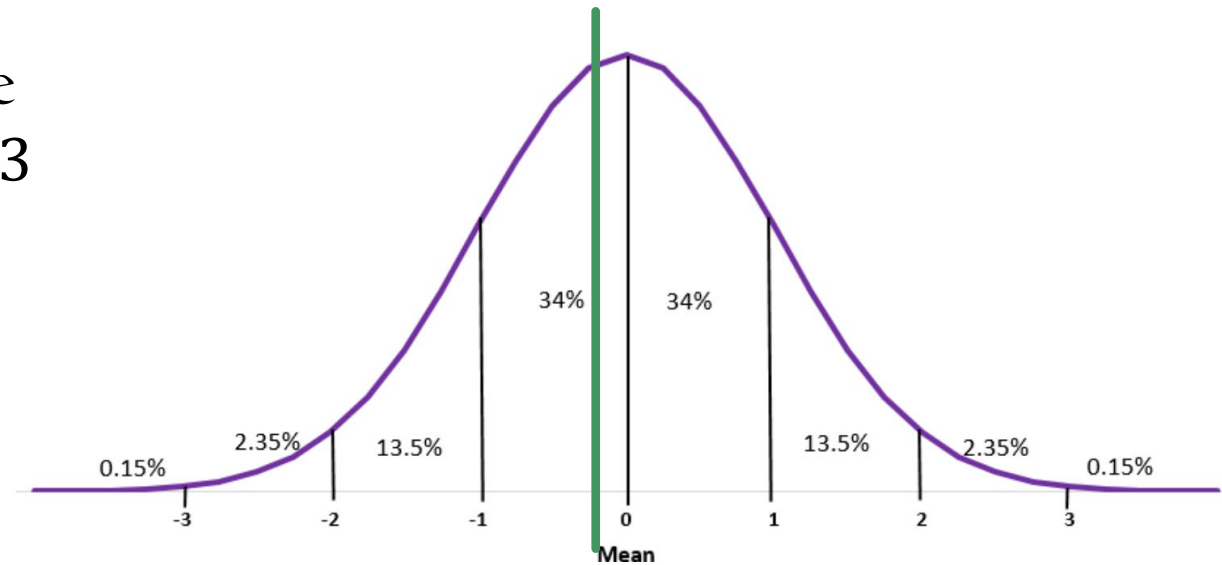| z	.00	.01	.02	.03	.04	.05
0.0	.5000	.5040	.5080	.5120	.5160	.519
0.1	.5398	.5438	.5478	.5517	.5557	.559
0.2	.5793	.5832	.5871	.5910	.5948	.598
0.3	.6179	.6217	.6255	.6293	.6331	.636
0.4	.6554	.6591	.6628	.6664	.6700	.673
0.5	.6915	.6950	.6985	.7019	.7054	.708
0.6	.7257	.7291	.7324	.7357	.7389	.742
0.7	.7580	.7611	.7642	.7673	.7704	.773
0.8	.7881	.7910	.7939	.7967	.7995	.802
0.9	.8159	.8186	.8212	.8238	.8264	.828
1.0	.8413	.8438	.8461	.8485	.8508	.853
1.1	.8643	.8665	.8686	.8708	.8729	.874
1.2	.8849	.8869	.8888	.8907	.8925	.894
1.3	.9032	.9049	.9066	.9082	.9099	.911
1.4	.9192	.9207	.9222	.9236	.9251	.926
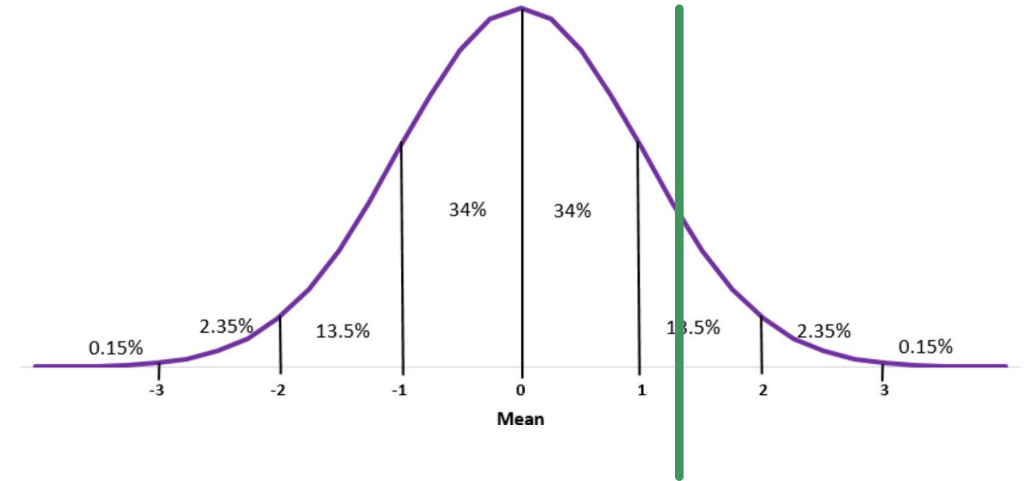
2. What is the probability of $z > -0.23$?

We use the value for $z = -0.23$ from the table of 0.409. However, we want greater than -0.23 so we have to subtract from 1. $1 - 0.409 = 0.591 = 59.1\%$

Table A Standard normal probabilities

| z	.00	.01	.02	.03	.04
−3.4	.0003	.0003	.0003	.0003	.0003
−3.3	.0005	.0005	.0005	.0004	.0004
•••					
−0.4	.3446	.3409	.3372	.3336	.3300
−0.3	.3821	.3783	.3745	.3707	.3669
−0.2	.4207	.4168	.4129	.4090	.4052
−0.1	.4602	.4562	.4522	.4483	.4443
−0.0	.5000	.4960	.4920	.4880	.4840

REMEMBER: this gives us the probability of LESS THAN the z-score!

3. What is $P(z < 2.13)$?

Table A   (Continued)

| z	.00	.01	.02	.03	.04
0.0	.5000	.5040	.5080	.5120	.5160
0.1	.5398	.5438	.5478	.5517	.5557
• • •					
2.0	.9772	.9778	.9783	.9788	.9793
2.1	.9821	.9826	.9830	.9834	.9838
2.2	.9861	.9864	.9868	.9871	.9875
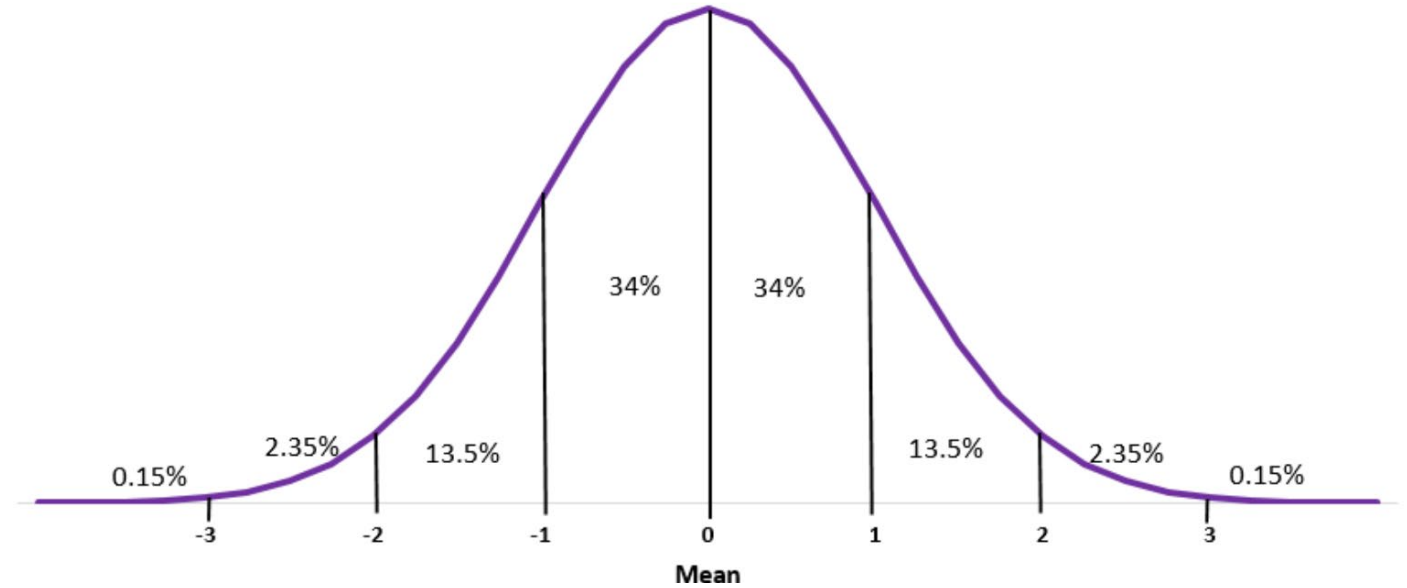
Since this is a positive z-score, we can use the value
for z = 2.13 directly from the table, and just express
it as a percentage: 0.9834 or 98.34%

What is the probability associated with a z-score between 1.2 and 2.31?

SOLUTION

To evaluate the probability of a value occurring within a given range, you need to find the probability of both the upper and lower values in the range, and subtract to find the difference.
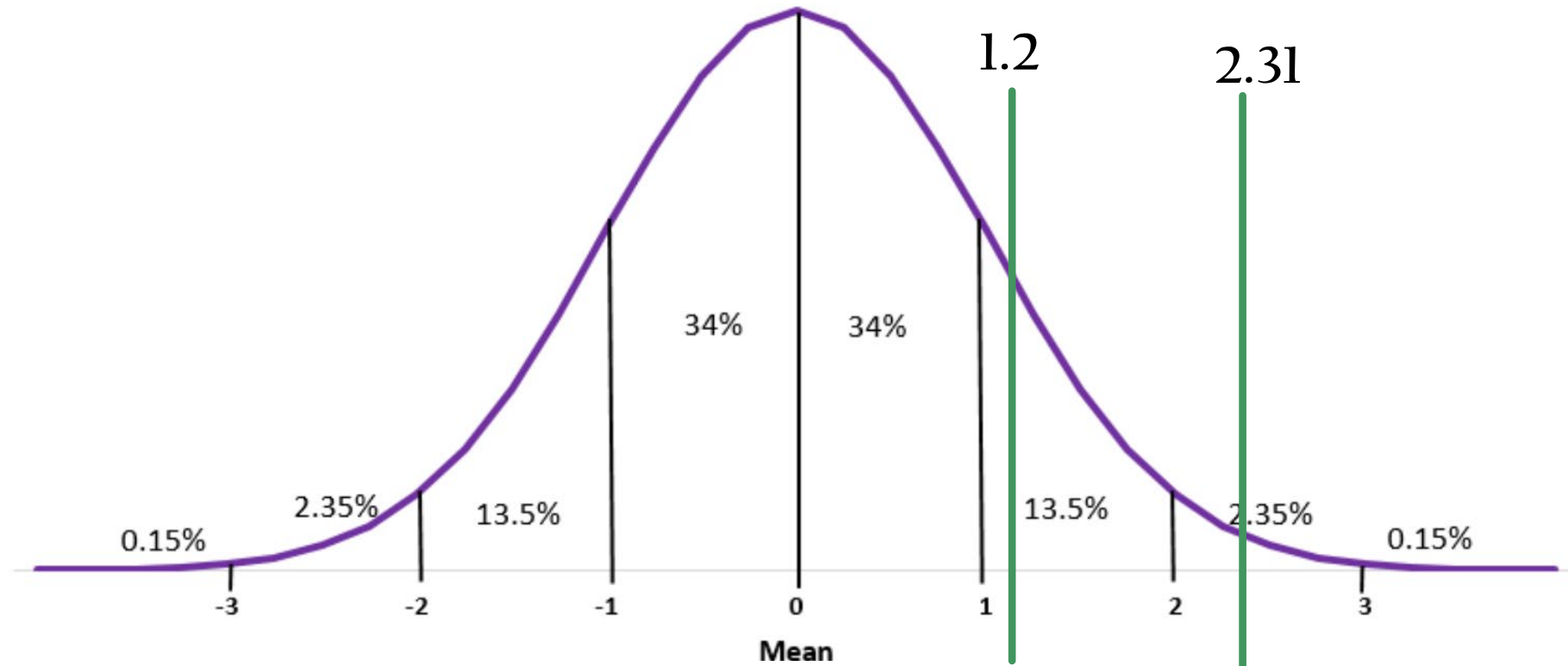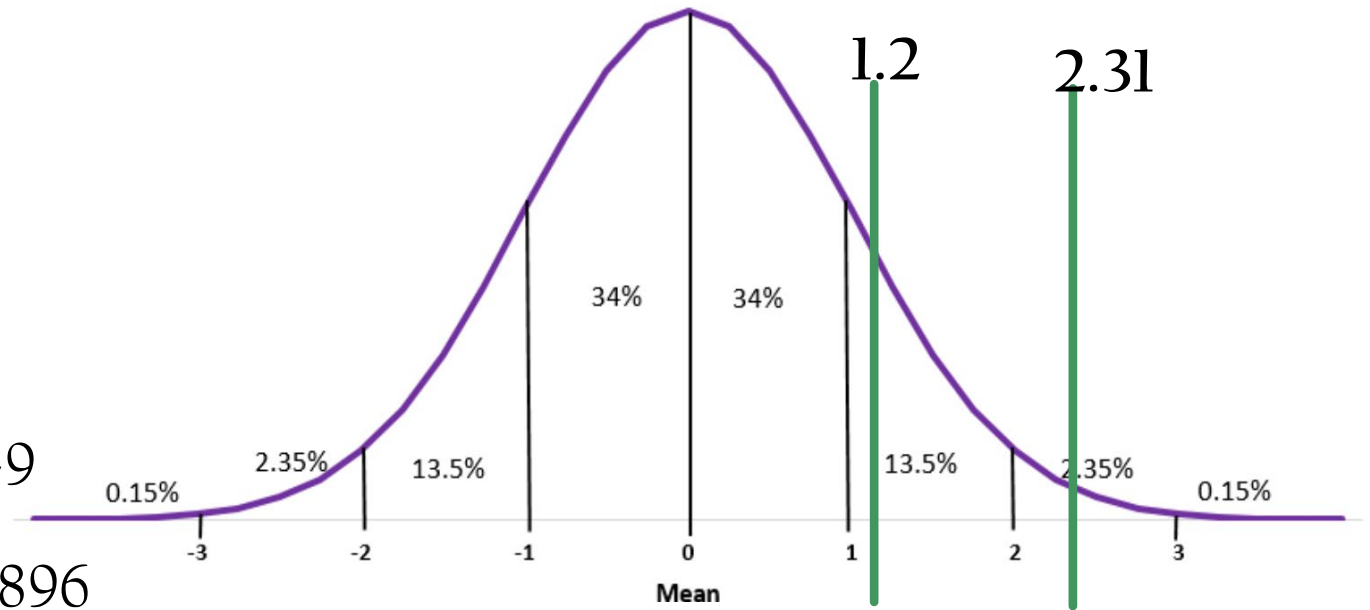
First find $z = 1.2$ on the z-table = 0.8849

Next, find $z = 2.31$ on the z-table = 0.9896

Since approximately 88.49% of all values are below z = 1.2 and approximately 98.96% of all values are below z = 2.31, there are $98.96\% - 88.49\% = 10.47\%$ of values between.

The probability of getting a value between z = 1.2 and 2.31 is 10.47%

On your own: What is the probability that a value with a z-score between -1.32 and 1.49 will occur in a normal distribution?

$$= P(z < 1.49) - P(z < -1.32)$$

$$= \quad .9319 \quad - \quad .0934$$

$$= \quad .8385$$

$$= \quad 83.85\%$$

Homework: See Teams for a worksheet